Chapter 21

EPR, Quantum Measurement and Bell's Theorem

The combination of the observation that there is a probability amplitude that is constructed by superimposing states by adding contributions of all paths, a wavelike property, and the interactions being instantaneous and stochastic leads to what are often interpreted as paradoxes in the quantum behavior of things. These ideas are treated under the heading of measurement theory.

Historically we could not do experiments on individual fundamental particles and could only deal with ensemble systems and the language was developed in that context. We are now entering an era in which we can manipulate individual fundamental systems. We are finding that all the rules that were developed in the ensemble language work for the fundamental systems.

In the following we deal with light and photons as our fundamental entities. This is a choice of convenience. Everything that I do here goes through for electrons. or any fundamental entity. The photon is a particularly simple system to deal with .

21.1 A Two Level System

In order to understand the essence of quantum mechanics and the measurement process in particular, lets study the simplest system possible. We will work with a system that has only two states and thus can appear as only a superposition of these two possible states. The double slit is an example. The light had to come from either slit one or slit two.

It turns out that light itself offers us an example of a two level system, the two polarizations of light. In the classical wave picture of light, the light is oscillations in the value of the electric field, $\vec{E}(x,t)$, and the magnetic field, $\vec{B}(x,t)$. Maxwell's equations determine that nature of the behavior of the electric and magnetic field. For light, the wavelike emissions, these equations require that the electric field be transverse to the direction of the motion of the light and that the magnetic field be perpendicular to both the electric field and the direction of propagation. Thus if we are given a direction for the light to travel, the electric field can only point in some direction in a plane, a two dimensional space.

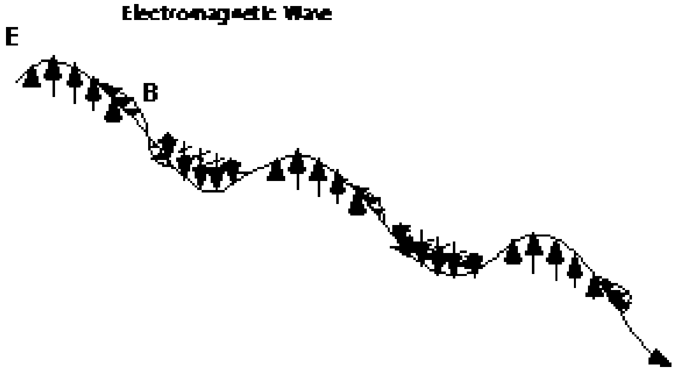

Figure 21.1: **Electric Wave** Light is high frequency, $5 \times 10^{-14}\ \frac{1}{\text{sec}}$, oscillations of the electric and magnetic fields that are traveling waves. Maxwell's equations require that the traveling waves have the electric and magnetic fields perpendicular to the one another and also to the direction of travel.

In one of our home experiments, we played with polarizers. These are sheets that absorb the light that has polarization transverse to a given direction and allows light polarized in the given direction to pass. In other words, light traveling in given direction comes in two varieties, let's say horizontal and vertical, which are at right angles with respect to each other.

If we had measured carefully in our home experiment with polarizers, we see that the light is a vector disturbance and that the vector amplitudes add.

If we insert an angled polarizer in the gap, we get some light.

For our purposes, the isolation of pure polarization states of light is

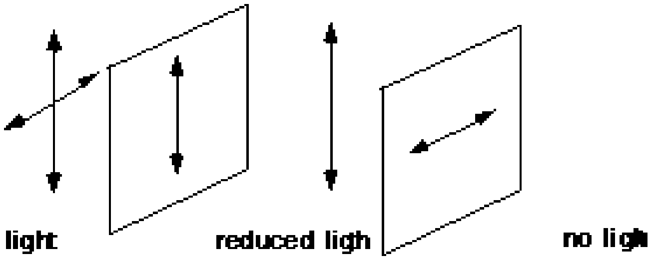

Figure 21.2: **Two Polarizers** Polarizers are a plastic material arranged in thin sheets that selectively absorbs light which is polarized in a given direction. Light from a thermal source is an incoherent jumble of light that is polarized in all directions. A polarizer placed in the beam removes all the light polarized in a given direction. In other words, a polarizer sheet has a hard direction, the direction of the polarized light that it absorbs and another direction perpendicular to that hard direction called easy that passes all light with that polarization. Unfortunately, polarizing sheets are identified by the easy direction. In the Figure above, thermal light enters from the left. The emergent light is now polarized in the easy direction. Since the thermal light has all polarizations, the intensity of the polarized light following the polarizer is half that of the incident beam. If a second polarizer is placed in the beam and oriented with the same orientation of the easy direction, the beam passes through it unaffected. On the contrary as shown above, if the second polarizer is rotated $\frac{\pi}{2}$ so that its easy direction is perpendicular to the first polarizer, no light passes.

better accomplished using birefringent materials[1]. The advantage is that neither of the polarization states of the light are absorbed. They are just separated.

All of these properties are simple to understand when we examine them from the wave picture. It becomes difficult when we combine with how that interactions happen. We will use the calcite crystal system to make a series of measurements on this two level system.

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[1] Birefringence is the process by which a single beam of light is split into two separated beams each with a given polarization. It was discovered in the 17^{th} century in calcite crystals and is now known to occur in many materials.

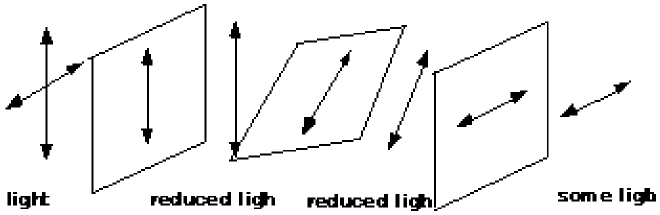

Figure 21.3: Three Polarizers

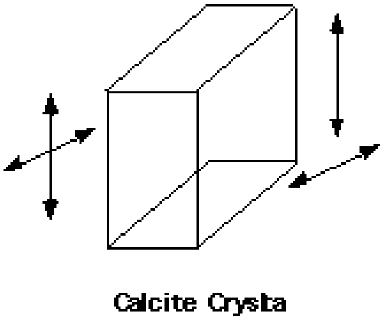

Figure 21.4: **Calcite Crystal** Calcite crystals have the property that they are birefringent. They separate a mixed polarization light beam into two spatially separated uniformly polarized beams of light.

21.2 More on polarized light as a two level system

We can use the calcite to divide a beam of light into the two polarization states:

The initial beam is a superposition of the two polarizations. The two emergent beams are in pure states of each polarization.

$$E_{in} = E_V + E_H = a\hat{E}_V + b\hat{E}_H \tag{21.1}$$

where if there are n photons per second coming in that are polarized equally between the two choices,

$$E_{in}^2 = n \tag{21.2}$$

and

$$a^2 + b^2 = n \tag{21.3}$$

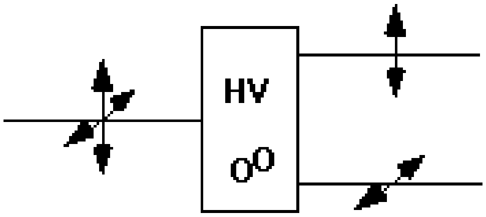

Figure 21.5: **Calcite Analyzer** The calcite crystal can cut so that the orientations of the two separated polarization states are along the vertical and horizontal directions. When this is the case, it is designated a 0^0 degree analyzer.

and I have defined $\hat{E}_V$ and $\hat{E}_H$ are the one photon state per second with the horizontal and vertical polarization, i. e. $\hat{E}_V^2 + \hat{E}_H^2 = 1$.

It is important to realize that you can have a calcite crystal that can separate out the two polarizations of light at different angles. In fact, any angle. Let's work with $\frac{\pi}{4}$ or 45^0.

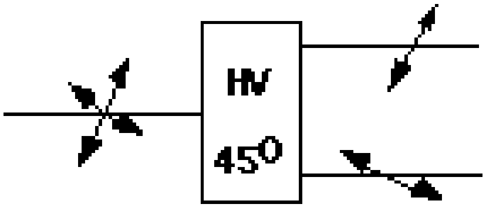

Figure 21.6: **A** 45^0 **Analyzer** By cutting the calcite crystal in different ways, the analyzer can be made to separate the beam into two beams with orthogonal polarizations in any angular orientations relative to the vertical.

What happens if you stack these things? As expected the second stage is consistent with the comment that the first stage in measuring the polarization has all the photons in it with the right polarization.

If you stack even more of the same type of polarizers you keep getting the same thing.

What happens if you mix angles?

Now you get light on all four channels.

What is the state of the light in the gap? In the upper leg, it is $a\hat{E}_V$ and the analyzer is at 45^0. Calling the two relevant directions $+45$ and -45, the state after the 45^0 analyzer is also described as

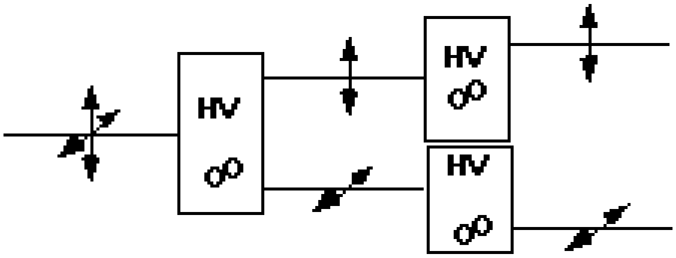

Figure 21.7: Stacked Analyzer

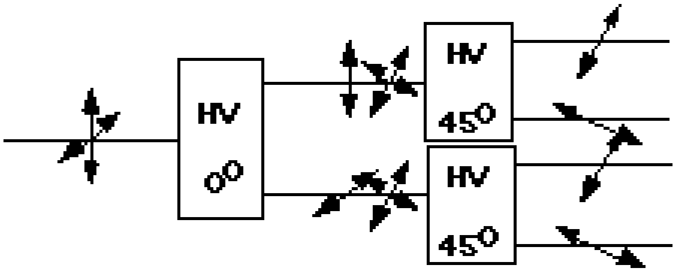

Figure 21.8: Stacked Turned Analyzer

$$a\hat{E}_V = c\hat{E}_{+45} + d\hat{E}_{-45} \tag{21.4}$$

with $a^2 = c^2 + d^2$ so that we have the correct number of photons. In other words, the state is a superposition of the +45 and −45 states. After the analyzer, we have c^2 photons in the upper most leg and d^2 photons in the second leg. What is the state of the photon in between the two analyzers. It is vertically polarized and it is a coherent mixture of +45 and −45. From our experience with the polarizers or if you like from the wave description of polarization for an arbitrary orientation, θ, we have

$$\hat{E}_V = \cos\theta\hat{E}_\theta + \sin\theta\hat{E}_{\perp\theta} \tag{21.5}$$

In our case, we have the $c = a\cos(45)$ and $d = a\sin(45)$.

You can also reconstruct the state of polarization that has been split. What happens if we look to see if the photon is in the upper branch of the middle legs or in the lower branch?

All of this leads to the question of what is the state of the photon. Again I have to emphasize our basic rule. Everything goes over all paths and has instantaneous local interactions that are stochastic.

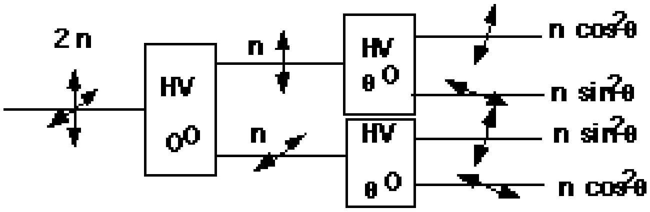

Figure 21.9: Intensity Analyzer

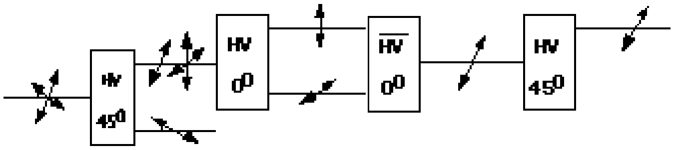

Figure 21.10: Tree of Analyzers

The state of the photon is in a superposition of polarization states. If we start with a vertically polarized beam and put it through a 45^0 analyzer, after the analyzer, it is half vertical and half horizontal. If we follow with a 0^0 analyzer, then we can then have, in one leg, a beam that is vertical. In the view of the individual photons that make up that last beam, when did they become vertical. Where they always vertical? If so where do we get the 45^0 beam come from. Did the vertical and horizontal photons interfere to produce the 45^0 beam. If we do this one photon at a time what happens? In our picture, we say that the effect of the analyzers is an interaction and we agree that interactions are stochastic and local in space and time. We say that the measurement changes the state,prepares the state. Another phrase that is used is that the superimposed state is collapsed into the measured state.

21.3 EPR and Bell's Theorem

Einstein always had trouble accepting the probability and field interpretation of quantum mechanics. In 1935, with B. Podolsky and N. Rosen, he proposed an experiment now called EPR that he thought made it clear that quantum mechanical predictions were implausible [EPR 1935]. The experiment involved the measurement of two separated photons that had been produced by a single decay. The apparatus is shown in Figure 21.12. At the center is a source[2]. At the end of the two decay pipes are two identical calcite crystals with photo detectors on each leg. If we modify the EPR apparatus by putting an analyzer with an arbitrary orientation on the end

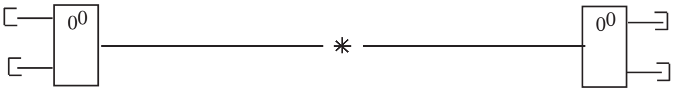

Figure 21.11: **EPR Apparatus** The apparatus discussed by Einstein Podolsky and Rosen

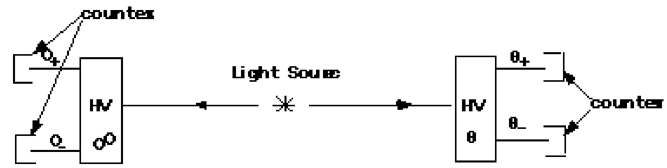

Figure 21.12: Modified EPR The apparatus discussed by Einstein Podolsky and Rosen

Using the properties of the analyzer, if we have n photons in this system we expect $\frac{n}{2}$ in the horizontal, 0_-, and $\frac{n}{2}$in the vertical, 0_+, counter of the left side of the apparatus. If the photon in the left is in 0_+, we know what the photon in right is. The problem is that we have reoriented the analyzer. On the left we get

$$\begin{aligned} n(0_+,\theta_+) &= \frac{1}{2} n \cos^2\theta \\ n(0_+,\theta_-) &= \frac{1}{2} n \sin^2\theta \\ n(0_-,\theta_+) &= \frac{1}{2} n \sin^2\theta \\ n(0_-,\theta_-) &= \frac{1}{2} n \cos^2\theta \end{aligned} \tag{21.6}$$

Define the correlation coefficient C

$$C \equiv \frac{\{n(0_+,\theta_+) + n(0_-,\theta_-) - n(0_+,\theta_-) - n(0_-,\theta_+)\}}{n} \tag{21.7}$$

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[2] One such source would be neutral pi zero mesons. The π^0s decay spontaneously into two identical photons traveling back to back. In addition, since the π^0s have no angular momentum the two photons must have opposite polarizations

If θ is zero then $C = 1$, they are correlated. If θ is $\frac{\pi}{2}$, $C = -1$, they are anticorrelated. Halfway, $\frac{\pi}{4}$, $C = 0$, they are not correlated at all. For us the correlation coefficient is

$$\begin{aligned} C &= \frac{\{\frac{1}{2}n\cos^2\theta + \frac{1}{2}n\cos^2\theta - \frac{1}{2}n\sin^2\theta - \frac{1}{2}n\sin^2\theta\}}{n} \\ &= \cos^2\theta - \sin^2\theta \\ &= \cos 2\theta \qquad (21.8) \\ & \qquad (21.9) \end{aligned}$$

Now consider three detections 0, ϕ, and θ. You can form lots of combinations,0_+, ϕ_-, θ_+. Make a table of random combinations

$$\begin{array}{ccc} 0 & \phi & \theta \\ + & - & + \\ - & + & - \\ + & + & + \\ \cdot & \cdot & \cdot \end{array} \qquad (21.10)$$

Let $n(\phi = +, \theta = -)$ be the number of sets with that configuration and so forth. Using figure 21.13, you can show that

$$n(0 = +, \phi = +) + n(\phi = -, \theta = +) \geq n(0 = +, \theta = +) \qquad (21.11)$$

In figure 21.13, the slices of the pie represent the number of triplets of each type. Note that $n(0 = +, \theta = +)$ is represented by sector AOC. Similarly, $n(\phi = -, \theta = +)$ is given by sector COE and $n(0 = +, \theta = +)$ by BOD. Clearly AOC + COE must be greater than or equal to BOD so it follows that

$$n(0 = +, \phi = +) + n(\phi = -, \theta = +) \geq n(0 = +, \theta = +) \qquad (21.12)$$

You do randomly different EPR experiments

Using the first set up we can measure $n(0_\pm, \phi_\pm)$ and so forth. We can measure all the parts of the inequality

$$n(0_-, \phi_+) + n(\phi_-, \theta_+) \geq n(0_-, \theta_+) \qquad (21.13)$$

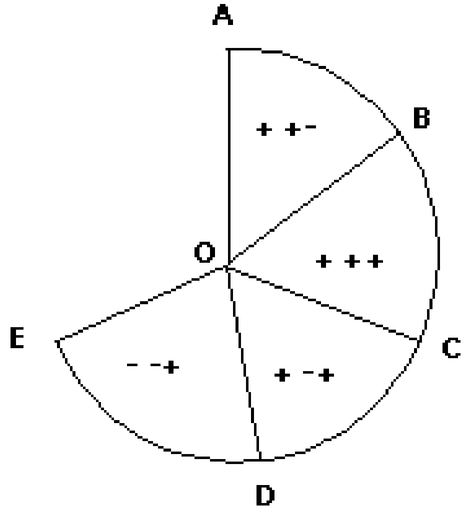

Figure 21.13: **Pie Chart of Probabilities** Pictorial representation of the table of randomly generated two valued choices for three things assignments. The area of the slices represent the fraction of the events with that assignment. Any probabilistic system has to be expressible in a chart of this kind.

Using equations 21.6

$$\cos^2\phi + \sin^2(\theta - \phi) \geq \cos^2\theta \tag{21.14}$$

Pick $\phi = 3\theta$. We will obtain

$$\cos^2 3\theta + \sin^2 2\theta - \cos^2\theta \geq 0 \tag{21.15}$$

But this should always be greater than zero. So quantum mechanics predicts things that can not happen with local random labels. The data follows the quantum mechanical prediction. Thus there can be no hidden variables theory consistent with these measurements.

Add material on incompatible measurements.

You are stuck with the measurement problems of quantum mechanics.

role of the observer

collapse of the wavepacket

Schrödinger's cat

many worlds

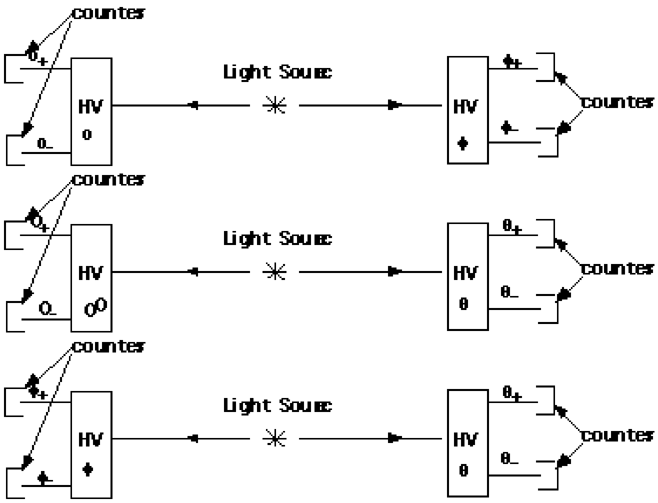

Figure 21.14: **Bell's configuration of Three EPR Apparatus** Bell suggested that three different EPR set ups with three different angles be run. Third of three configurations used in Bell's inequality. This apparatus measures the correlation between the angle ϕ and θ.

21.3.1 What is a particle and what is the field ?

We now know quite a bit about the photon. This is the object that carries the energy and momentum of the electromagnetic field. Yet we know that electromagnetism has field properties. How do we reconcile these observations. We should realize that we observe the field nature when there are many photons present, i. e. in cases in which the energy is many times $\hbar\omega$. How do we make a quantum theory of the electromagnetic field?

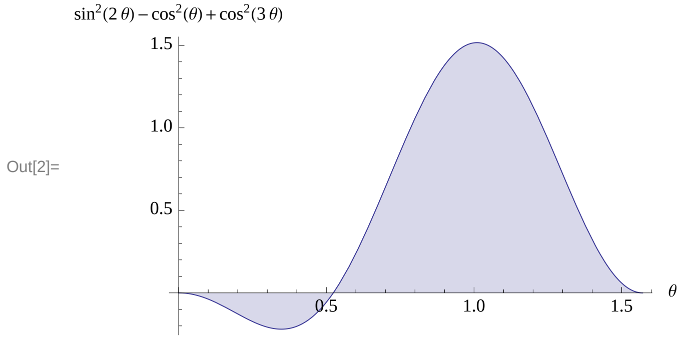

Figure 21.15: **Bell's Inequality**, Equation 21.13, which must be satisfied by any local theory of probabilistic transmission is not satisfied by quantum mechanics. When the appropriate amplitudes produced by a quantum mechanical system is used in the inequality it predicts that $\cos^2 3\theta + \sin^2 2\theta - \cos^2 \theta$ must always be greater than zero. As can be seen above, for angles less than about 0.5 radians it does not satisfy the inequality. Experiment agrees with the quantum mechanical results.